Double Elliptic Systems: Problems and Perspectives [†]

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Abstract

This talk presents a list of problems related to the double-elliptic (Dell) integrable systems with elliptic dependence on both momenta and coordinates. As expected, in the framework of Seiberg-Witten (SW) theory the recently discovered explicit self-dual family of 2-particle Dell Hamiltonians provides a perturbative period matrix which is a logarithm of the ratio of the (momentum-space) theta-functions.

Double-elliptic integrable systems for a long time remain among the most mysterious objects of string and mathematical physics: their existence is implied by many branches of science, while they have been never explicitly constructed.

Recently we have finally obtained [3, 4] an explicit expression for the Dell Hamiltonian in the simplest case of $SU(2)$ (two-particle model) and suggested a direct way to find – at least in a somewhat transcendental form – the Hamiltonians for all the $SU(N)$. It makes sense now to recollect all the subjects where the Dell systems were expected to manifest themselves and begin the program of putting things together. This note presents just a list of topics and open problems with minimal comments.

1 Dell system

According to [3, 4] the double elliptic system of 2 particles in the center-of-mass frame is described by the Hamiltonian:

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$$H(p, q|k, \tilde{k}) = \alpha(q|\tilde{k}) \mathrm{cn} \left(p\beta(q|k, \tilde{k}) \middle| \frac{k\alpha(q|\tilde{k})}{\beta(q|k, \tilde{k})} \right). \tag{1}$$

Here

$$\alpha^2(q|\tilde{k}) \equiv 1 - \frac{2g^2}{\mathrm{sn}^2(q|\tilde{k})}, \quad \beta^2(q|k, \tilde{k}) \equiv k'^2 + k^2\alpha^2(q|\tilde{k}) = 1 - \frac{2g^2 k^2}{\mathrm{sn}^2(q|\tilde{k})} \tag{2}$$

sn and cn are the standard Jacobi functions. The model is parametrized by two independent (momentum and coordinate) *bare* elliptic curves with elliptic moduli k and $\tilde{k}$ ($k' = \sqrt{1 - k^2}$ and $\tilde{k}' = \sqrt{1 - \tilde{k}^2}$ are the complimentary elliptic moduli). The *full* spectral curve

$$H(\eta, \xi|k, \tilde{k}) = u \quad (= \mathrm{cn}(Q|k)) \tag{3}$$

is characterized by the effective elliptic moduli

$$k_{eff} = \frac{k\alpha(q|\tilde{k})}{\beta(q|k, \tilde{k})}, \quad \tilde{k}_{eff} = \frac{\tilde{k}\alpha(q|k)}{\beta(q|\tilde{k}, k)} \tag{4}$$

Coordinate-momentum duality [5] interchanges $k \leftrightarrow \tilde{k}$, $k_{eff} \leftrightarrow \tilde{k}_{eff}$ (and $q, p \leftrightarrow Q, P$).

In general $SU(N)$ case, the model describes an interplay between the four tori: the two bare elliptic curves and two effective Jacobians of complex dimension $g = N - 1$.

The $SU(2)$ Hamiltonian (1) has a rather nice form in terms of the p, q variables or their duals P, Q. However, the general construction of refs.[3, 4] for $SU(N)$ describes Hamiltonians as ratios of genus g theta-functions which depend on another kind of canonical variables – angle-action variables p_i^{Jac}, a_i. The *flat* moduli a_i play the central role in the SW theory, while Q_i are rather generalizations of the algebraic moduli; in the most familiar case of the Toda chain the spectral curve is

$$w + \frac{\Lambda_{QCD}^{2N}}{w} = \prod_{i=1}^{N}(\lambda - Q_i)$$

while $a_i = \oint_{A_i} \lambda \frac{dw}{w}$. For our Dell system these variables a_i can be calculated as A-periods

$$a_i = \oint_{A_i} dS \tag{5}$$

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of the generating differential $dS = \eta d\xi$ on the spectral curve (3), while its B-periods define the prepotential $\mathcal{F}(a)$ through

$$\frac{\partial \mathcal{F}}{\partial a_i} = \oint_{B_i} dS \qquad (6)$$

It is an interesting open problem to express the Hamiltonians in terms of P_i and Q_i, perhaps, they can acquire a more transparent form, like it happens for $SU(2)$. Moreover, in [3, 4] the Hamiltonians for $N > 2$ were found only in the limit $\tilde{k} = 0$ (elliptic-trigonometric model, the dual of elliptic Ruijsenaars): elliptization of coordinates (switching on $k \neq 1$) should be straightforward (the real problem have been to elliptize the momentum dependence), but still remains to be performed. Another option is to switch to the "separated" variables $\mathcal{P}_i$, $\mathcal{Q}_i$ [6] such that $\oint_{A_j} \mathcal{P}_i d\mathcal{Q}_i = a_i \delta_{ij}$ (while generically $a_i = \oint_{A_j} \sum_i P_i dQ_i$). Generically, they are different from P_i, Q_i but coincide with P, Q in the case of $SU(2)$.

To conclude our description of the $SU(2)$ Dell system, we calculate the perturbative prepotential (*i.e.* the leading order of the expansion in powers of $\tilde{k}$ when the bare spectral torus degenerates into sphere) that allows one to establish the identification with physical theories. In the forthcoming calculation we closely follow the line of [7] where the very similar calculation has been done for the Ruijsenaars model.

When $\tilde{k} \to 0$, $\text{sn}(q|\tilde{k})$ degenerates into the ordinary sine. For further convenience, we shall parametrize the coupling constant $2g^2 \equiv \text{sn}^2(\epsilon|k)$. Now the spectral curve (3) acquires the form

$$\alpha(\xi) \equiv \sqrt{1 - \frac{\text{sn}^2(\epsilon|k)}{\sin^2 \xi}} = \frac{u}{\text{cn}\left(\eta\beta\Big|k_{eff}\right)} \qquad (7)$$

Here the variable ξ lives in the cylinder produced after degenerating the bare coordinate torus. So does the variable $x = 1/\sin^2 \xi$. Note that the A-period of the dressed torus shrinks on the sphere to a contour around $x = 0$. Similarly, B-period can be taken as a contour passing from $x = 0$ to $x = 1$ and back.

The next step is to calculate variation of the generating differential $dS = \eta d\xi$ w.r.t. the modulus u in order to obtain a holomorphic differential:

$$dv = \left(-i\text{sn}(\epsilon|k)\sqrt{k'^2 + k^2 u^2}\right)^{-1} \frac{dx}{x\sqrt{(x-1)(U^2 - x)}} \qquad (8)$$

where $U^2 \equiv \frac{1-u^2}{\text{sn}^2(\epsilon|k)}$. Since

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$$\frac{\partial a}{\partial u} = \oint_{x=0} \frac{\partial dS}{\partial u} = \oint_{x=0} dv = -\frac{1}{\sqrt{(1-u^2)(k'^2 + k^2u^2)}} \qquad (9)$$

we deduce that $u = \text{cn}(a|k)$ and $U = \frac{\text{sn}(a|k)}{\text{sn}(\epsilon|k)}$. The ratio of the B- and A-periods of dv gives the period matrix

$$T = \frac{U}{\pi} \int_0^1 \frac{dx}{x\sqrt{(x-1)(U^2-x)}} = -\frac{1}{i\pi} \lim_{\kappa \to 0} \left(\log \frac{\kappa}{4} \right) + \frac{1}{i\pi} \log \frac{U^2}{1-U^2} \qquad (10)$$

where κ is a small-x cut-off. The U dependent part of this integral is finite and can be considered as the "true" perturbative correction, while the divergent part just renormalizes the bare "classical" coupling constant τ, *i.e.* classical part of the prepotential (see [7] for further details). Therefore, the perturbative period matrix is finally

$$T_{finite} = \frac{i}{\pi} \log \frac{\text{sn}^2(\epsilon|k) - \text{sn}^2(a|k)}{\text{sn}^2(a|k)} + \ const \ \longrightarrow \ \frac{i}{\pi} \log \frac{\theta_1(a+\epsilon)\theta_1(a-\epsilon)}{\theta_1^2(a)} \qquad (11)$$

and the perturbative prepotential is the elliptic tri-logarithm (cf. [8]). Remarkably, it lives on the *bare* momentum torus, while the modulus of the perturbative curve (7) is the dressed one.

2 Dell systems in the context of modern theory

Roughly, one can divide the interest to the Dell systems into two flows: coming from string physics and from the theory of integrable systems *per se*. In Fig.1 these subjects are listed at the l.h.s. and the r.h.s. respectively.

2.1 Physics

In physics, renewed interest to integrable systems is due to the discovery of their role in quantum field theory: exact (non-perturbative) effective actions usually exhibit integrable properties, if considered as functions of boundary conditions (coordinates in the moduli space of vacua) and coupling constants [9, 10, 11]. The most primitive classical multiparticle integrable systems (well reduced KP/Toda-lattice models) emerge in this way [2] in the study of vacuum dependencies of the low-energy effective actions of $N = 2$ SUSY Yang-Mills theories [1]. The table of known relations is shown in Fig.2. When SUSY is broken down to $N = 1$ or $N = 0$, continuous moduli normally disappear and only particular
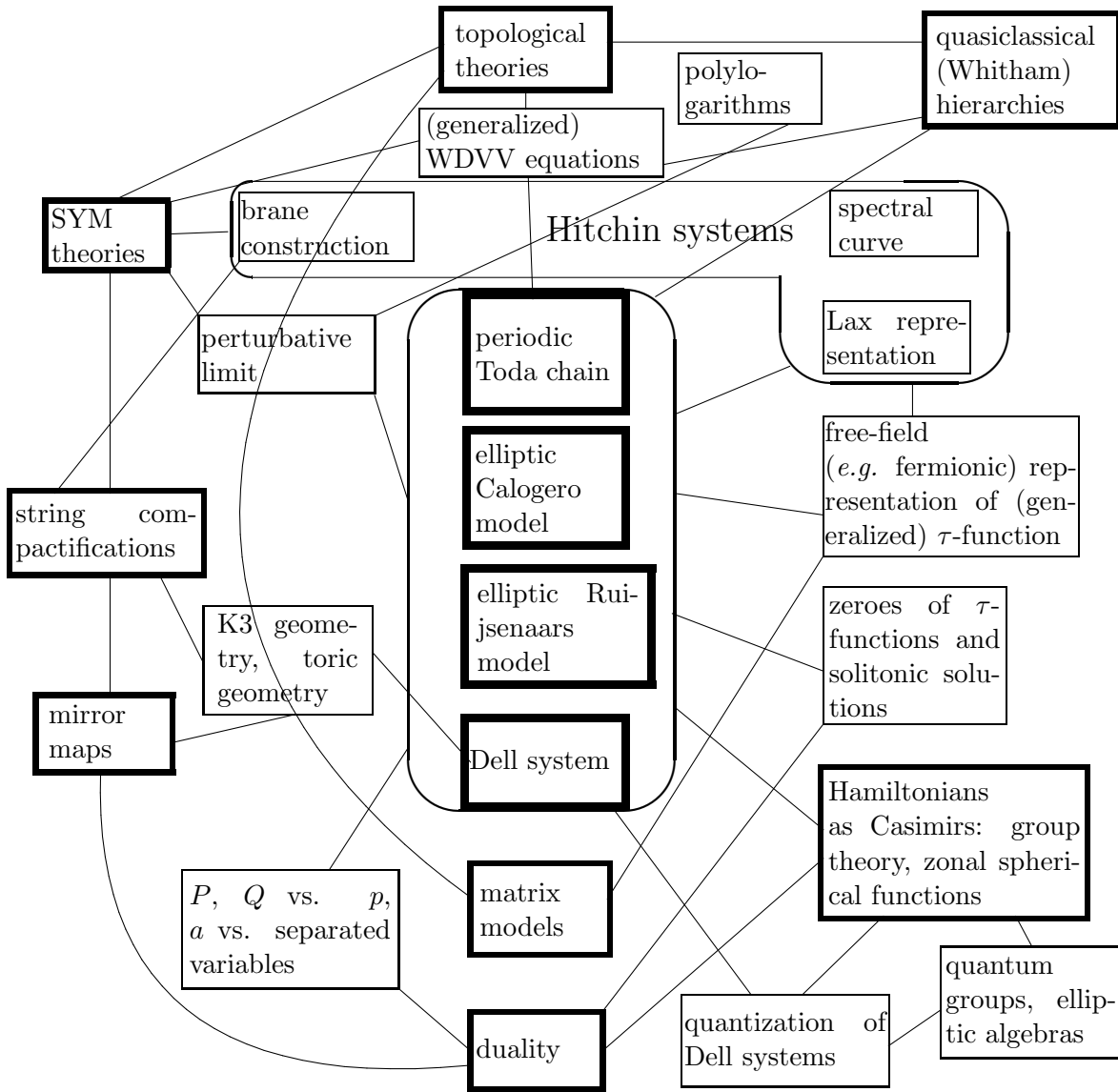
topological
theories

quasiclassical
(Whitham)
hierarchies

polylo-
garithms

(generalized)
WDVV equations

brane
construction

Hitchin systems

spectral
curve

SYM
theories

perturbative
limit

periodic
Toda chain

Lax repre-
sentation

elliptic
Calogero
model

free-field
(*e.g.* fermionic) rep-
resentation of (gen-
eralized) τ-function

string com-
pactifications

K3 geome-
try, toric
geometry

elliptic Rui-
jsenaars
model

zeroes of τ-
functions and
solitonic solu-
tions

mirror
maps

Dell system

Hamiltonians
as Casimirs: group
theory, zonal spheri-
cal functions

P, Q vs. p,
a vs. separated
variables

matrix
models

quantum
groups, ellip-
tic algebras

duality

quantization of
Dell systems

Figure 1: Around double elliptic systems

SUSY physical theory	Pure gauge SYM theory, gauge group G	SYM theory with fund. matter	SYM theory with adj. matter
4d	inhomogeneous periodic Toda chain for the dual affine $\hat{G}^\vee$ (*non-periodic Toda chain*)	elliptic Calogero model (*trigonometric Calogero model*)	inhomogeneous periodic XXX spin chain (*non-periodic chain*)
5d	periodic relativistic Toda chain (*non-periodic chain*)	elliptic Ruijsenaars model (*trigonometric Ruijsenaars*)	periodic XXZ spin chain (*non-periodic chain*)
6d	periodic "Elliptic" Toda chain (*non-periodic chain*)	Dell system (*dual to elliptic Ruijsenaars, elliptic-trig.*)	periodic XYZ (elliptic) spin chain (*non-periodic chain*)

Figure 2: SUSY gauge theories $\Longleftrightarrow$ integrable systems correspondence. The perturbative limit is marked by the italic font (in parenthesis).

spectral curves survive from entire Seiberg-Witten (SW) families. However, one can still extract the dynamical information from SW theory, restricting the answers for correlation functions to given points in the moduli space.

The most interesting distinctions between various models in Fig.2 are:

(i) UV-finite *vs.* UV-infinite (where dimensional transmutation occurs) models. In the language of integrable systems the difference is in the *bare* spectral surface (complex curve): it may be elliptic one (torus) for the UV-finite models*, while it degenerates into a punctured sphere

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*The situation is still unclear in application to the case of fundamental matter with $N_f = 2N_c$. In existing formulation for spin chains the bare coupling constant appears rather as a twist in gluing the ends of the chain together [16] (this parameter occurs only when $N_f = 2N_c$) and is not immediately identified as a modulus of a *bare* elliptic curve. This problem is a fragment of a more general puzzle: spin chains have not been described as Hitchin systems; only the "2×2" Lax representation is known, while its "dual" $N_c \times N_c$

in UV-infinite cases.

(ii) Matter in adjoint *vs.* fundamental representations of the gauge group. Matter in adjoint representation can be described in terms of a larger pure SYM model, either with higher SUSY or in higher space-time dimension. Thus models with adjoint matter form a hierarchy, naturally associated with the hierarchy of integrable models *Toda chain* $\hookrightarrow$ *Calogero* $\hookrightarrow$ *Ruijsenaars* $\hookrightarrow$ *Dell* [2, 12, 13, 14, 7, 3]. Similarly, the models with fundamental matter are related to the hierarchy of spin chains originated from the Toda chain: *Toda chain* $\hookrightarrow$ *XXX* $\hookrightarrow$ *XXZ* $\hookrightarrow$ *XYZ* [15, 16, 8].

(iii) SYM models in different dimensions. Actually, Yang-Mills fields play a distinguished role (representing the universality class of conformal invariant models, perhaps, broken by the dimensional transmutation) only in 4 and 5 dimensions. Generically, in $d = 2k$ their role is taken by theories of $(k-1)$-forms. The relevant model for odd $d = 2k+1$ can be described both in terms of $(k-1)$- and k-forms (like $3d$ Chern-Simons model is expressed both in terms of gauge vectors and in terms of the scalar fields of $2d$ WZNW model). Integrable systems relevant for the description of vacua of $d = 4$ and $d = 5$ models are respectively the Calogero and Ruijsenaars ones (which possess the ordinary Toda chain and "relativistic Toda chain" as Inosemtsev's limits [17]).

SYM models can be embedded into the more general context of string models – and, thus, into the entire framework of the future string theory – in two ways: through stringy compactifications [18] and through non-trivial brane configurations [19]. The both approaches can be dealt with in different ways, starting from different superstringy models in $D = 10, 11, 12$ dimensions. Actually, the relevant brane configurations involve branes with some compact dimensions and, therefore, this is actually also the story about compactifications, only from lower $D = 6, 7, 8$. The compact dimensions are interpreted as complex spectral curves within the brane approach [20] or their non-compact $3_C d$ mirrors [21] – the singular Calabi-Yau-like manifolds – in string compactifications. The Lax operators are provided by non-trivial solutions of the scalar-fields equations of motion on the bare spectral curve [22] (in other words, an attempt to compactify on a *bare* spectral curve ends up with the dynamical formation of a more sophisticated *full* spectral curve, *i.e.* the bare curve gets "spontaneously" fibrated).

As usual, when compactification size is large as compared to the Plankian (stringy) scale, one can consider the effective low-energy theory, neglecting the tower of stringy massive states and gravitational

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one is not yet available.

interactions. This effective theory includes massless gauge fields and their relatively light companions with masses inversely proportional to the compactification size, which can be identified with the gauge fields acquiring their masses through the Higgs mechanism and with the solitons/monopoles associated with strings winding around the compact dimensions. The description of such effective theory is in terms of *prepotentials* that celebrates a lot of properties familiar from the original studies of pure topological theories (where have been neglected the possibility of the light excitations to move). These properties include the identification [2, 14, 25] of prepotentials as quasiclassical (Whitham) τ-functions [23, 24] and peculiar equations, of which the (generalized) WDVV equations [26, 24, 27] are the best known example.

In the prepotential, the contributions of particles and solitons/monopoles (dyons) sharing the same mass scale, are still distinguishable, because of different dependencies on the bare coupling constant, *i.e.* on the modulus τ of the bare coordinate elliptic curve (in the UV-finite case) or on the Λ_{QCD} parameter (emerging after dimensional transmutation in UV-infinite cases). In the limit $\tau \to i\infty$ ($\Lambda_{QCD} \to 0$), the solitons/monopoles do not contribute and the prepotential reduces to the "perturbative" one, describing the spectrum of non-interacting *particles*. It is immediately given by the SUSY Coleman-Weinberg formula [27]:

$$\mathcal{F}_{pert}(a) = \sum_{\text{reps } R,i} (-)^F \text{Tr}_R (a + M_i)^2 \log(a + M_i) \tag{12}$$

SW theory (actually, the identification of appropriate integrable system) can be used to construct the non-perturbative prepotential, describing the mass spectrum of all the "light" (non-stringy) excitations (including solitons/monopoles). Switching on Whitham times [25] presumably allows one to extract some correlation functions in the "light" sector.

In accordance with the general logic of string theory [9], every model of quantum field theory possesses different descriptions in terms of different perturbative Lagrangians (referring to different phases). Thus, the models looking different perturbatively can be, in fact, identical non-perturbatively. If many enough couplings are allowed, every particular model gets embedded into a unique and universal stringy "theory of everything". At the present stage of knowledge we just begin to observe particular traces of this general phenomenon in the form of sporadic "duality" identities between particular string models [28]. A further restricted class of dualities in application to $N = 2$ SYM can be identified with simple canonical transformations. This is the coordinate-momentum duality of integrable systems [5].
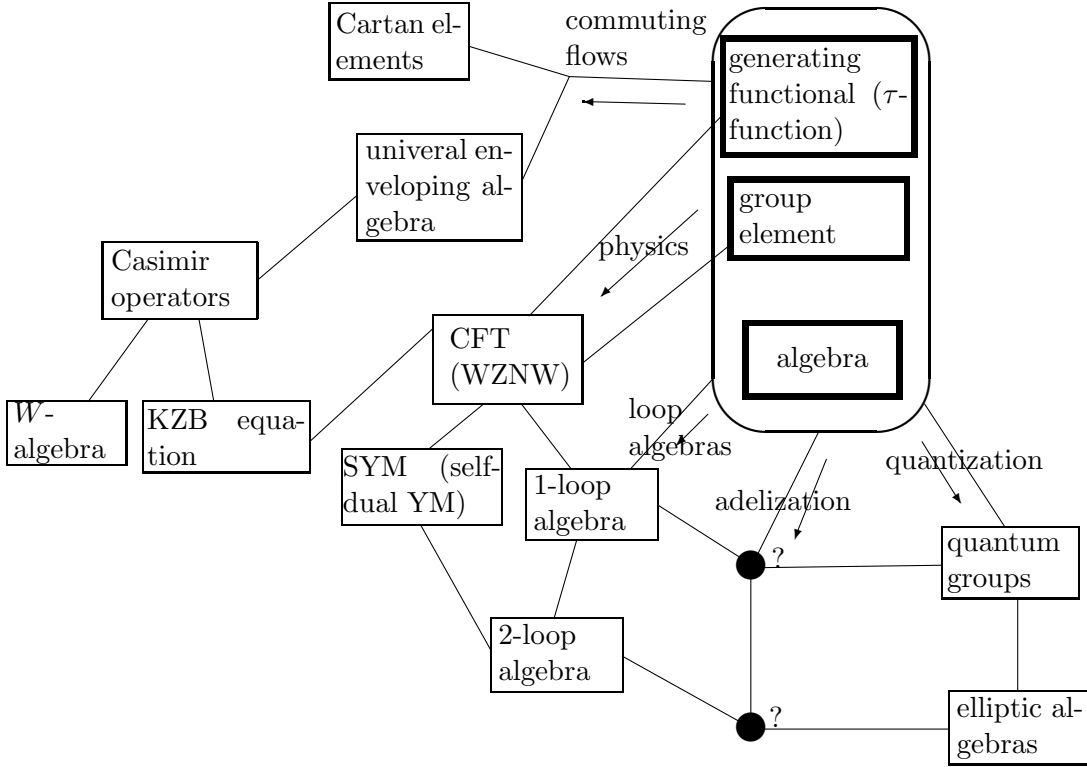
Figure 3: Some relevant branches of group theory

2.2 Integrability theory

Integrability theory studies commuting Hamiltonian flows. A minor generalization allows the flows to form a closed non-abelian algebra. Thus, integrability theory is naturally a branch of group theory (see Fig.3), studying the group element of a (Lie, quantum, elliptic, ...) algebra as a (classical, quantum, elliptic(?), ...) multi-time evolution operator. The central object in the theory is the collection of matrix elements of the evolution operator in a given representation assembled into a generating function called (generalized) τ-function [10, 29] of the corresponding algebra. Morphisms of representations (like decomposition of tensor products $R_i \otimes R_j = \oplus_k N_{ij}^k R_k$) induce Hirota-like relations between the τ-functions in different representations [30], which can be rewritten as a hierarchy of (differential, finite-difference, elliptic(?)...) equations.

From this perspective, integrability is not so much concentrated on

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Hamiltonians and their commutativity, but the really important issue becomes links with representation theory. Still, if one is interested in *commuting* flows, the two natural places for the commuting flows to emerge are in studies of the Cartan-like subalgebras and of the Casimir operators. If concentrated on Hamiltonians, people often distinguish between the classical Hamiltonian equations of motion, $\frac{\partial p_i}{\partial t_n} = -\frac{\partial H_n}{\partial q_i}$, $\frac{\partial q_i}{\partial t_n} = \frac{\partial H_n}{\partial p_i}$ and the Schrödinger equations $i\frac{\partial \Psi}{\partial t_n} = \hat{H}_n \Psi$. Then, a way from group theory to integrable Hamiltonians is through restricting Casimir operators to particular orbits (the theory of Hamiltonian reductions [31]). This gives rise to Laplace-like (differential and shift) operators together with their eigenfunctions (wave functions or zonal spherical functions). The eigenfunctions being represented as particular matrix elements are contained in the (generalized) τ-function. The problem here is to extend the formalism to more sophisticated groups, like quantum affine [32] and elliptic [33] algebras. Only at affine elliptic level one expects the Dell hamiltonians to appear. Note that the coordinate-momentum duality acts on the zonal spherical functions, interchanging the arguments and eigenvalues [34].

An *a priori* different *quantization* of integrability comes from a different place: classical and quantum integrability can be associated with continuous and discrete-time evolutions and thus related to classical and quantum groups respectively. The most intriguing from the group theory point of view is the quasiclassical (Whitham) integrability [23, 35]. Further generalization to the case of a two-component Plank constant imply a sort of elliptic quantization associated with emerging elliptic algebras [33]. Their full description will hopefully shed light also on the somewhat mysterious "p-adic dimension", empirically observed, among other places, in the properties of McDonald polynomials [36].

On the other hand, elliptic algebras are naturally expected to appear [37] in the study of loop algebras (*cf.* with Kazhdan-Lüstig framework, see also [38]): given an algebra, one can consider the corresponding 1-loop, 2-loop etc algebras (mapping from circles, Riemann surfaces etc into the algebra). At every level, an interesting new deformation shows up. At the 1-loop level, the central extensions give rise to affine (Kac-Moody) algebras. At the 2-loop level, a Moyal bracket (quantum deformation) should occur etc.

The most adequate tool for group theory investigations is provided by quantum field theory (geometrical quantization). The hidden group-theoretical structures manifest themselves by high quality of the approximation, provided by the transformation to free fields (Darboux variables). "High quality" normally means that the corrections are con-

centrated in co-dimension one or higher (in other words, the original theory can be substituted by a mixture of the free-field ones with various non-trivial boundary conditions, *i.e.* with non-trivial theories on the boundaries). To put it differently, the functional integral for d-dimensional theory in compact space-time can be reduced to a collection of d-dimensional Green functions (correlators of free fields) and non-trivial functional integral over fields on some hypersurfaces. This phenomenon is well studied (and partly understood) in the theory of $2d$ conformal models: the WZNW model can be represented in terms of free fields with insertions of additional "screenings" realized as contour integrals [39]. Another (not unrelated) example is provided by the old theory of hierarchy of anomalies [40]. The hierarchy relates, for example, the Donaldson theory in $4d$ (3-loop level), Chern-Simons theory in $3d$ (2-loop level; co-dimension one), WZNW theory in $2d$ (1-loop level; co-dimension two) etc.

The free field formulation is intensively used in the theory of conventional KP/Toda-lattice τ-functions [41] (of which the Dell-Ruijsenaars-Calogero-Toda-chain family is a very specific reduction) associated with the affine (Kac-Moody) algebras of level $k = 1$ and also with the *eigenvalue* matrix models [10, 11]. The next step will be made when more general "non-abelian" or "non-eigenvalue" τ-functions associated with the level $k \neq 1$ emerge from the future studies of effective actions (generating functions of correlators) of the $2d$ WZNW model and $3d$ Chern-Simons theory. However, the real breakthrough, when the interrelation between different fashionable subjects is to become transparent, is expected after the τ-functions of double-loop algebras (like that of area-preserving diffeomorphisms and/or the ones implicit in emerging the deformation quantization theory [42]) get enough attention.

In algebraic geometry, instead of studying representation theory of groups *per se*, in Tanaka-Krein style, one considers particular sophisticated representations, like those in bundles/sheaves over complex curves and surfaces, thus mixing the group theory and algebraic geometry data. This line of research is nowadays labeled as Hitchin [43] or SW theory and it provides the most of currently intriguing links to non-perturbative dynamics of quantum physical systems. Remarkably, it is enough to input just very simple *bare* spectral curves, like punctured Riemann sphere or torus, in order to get from the geometrical quantization technique some very non-trivial families of the *full* spectral curves (dynamics induced by group theory "spontaneously" transforms trivial curves into highly non-trivial ones) and somewhat sophisticated prepotentials belonging to the fashionable classes of special functions (like

polylogarithims and their generalizations). The most interesting issue here is that the prepotential turns out to be a quasiclassical τ-function [2, 25] and, therefore, it can be obtained in two very different ways: from SW/Hitchin theory, where, at least, the *bare* spectral curves are used as input, and from quasiclassical limits of ordinary (generalized) τ-functions built directly from affine quantum groups, where the only naive origin for a *curve* to appear is due to dealing with loop algebras. This situation echoes the general one with the quasiclassical quantization of stringy theory. Quasiclassical approximation always depends on two things: dynamics itself (which is dictated by symmetry principles and group theory) and the choice of "vacuum" (which usually breaks a lot of symmetries and is naturally an object of algebraic geometry nature). Thus, quasiclassical dynamics can be described in two languages: in terms of the full quantum dynamics (*i.e.* – at the end of the day – in terms of ordinary generalized τ-functions) and in terms of deformations of the vacuum (*i.e.* in terms of deformation theory, say, of a Hodge style one). In the last description, one expects also additional (duality) relations between quasiclassical theories nearby different vacua to occur, which reflects the existence of the full quantum dynamics *common* for all the vacua. One should keep in mind that, in the situation with many deformation parameters, the same words can be said about "the quasiclassical approximation" w.r.t. any of the parameters. The future string theory is expected to unify in this manner all quantum field models, dealing with them as perturbative (= quasiclassical) approximations to its different phases (= vacua) with the Plank constant substituted by the inverse Plank mass as the deformation parameter... Coming back to SW theory, what really remains to discover within this context is the rich structure of the "vacua" (associated with the variety of complex spectral curves) in the geometrical quantization of affine quantum groups.

3   Open problems involving Dell systems

Actually, most of the presentation of the previous section describe the open problems: every line calls for further clarification and details. Still, it makes sense to list a number of lower scale particular problems, directly involving the Dell systems.

Complete construction of Dell integrable systems
- Explicit construction of Dell Hamiltonians for groups larger than $SU(2)$

- Commutativity of the Hamiltonians

- Quantization of the Hamiltonians

- Inozemtsev's limit (for coordinate torus) [17, 13] of the Dell system: "elliptic Toda chain" [44]

Identification of the relevant gauge theory

- Perturbative prepotential

- Relation to spin chains

- String compactifications

- Torus fibered over torus, particular K3 surfaces

- Relevant brane configurations and gauge theory on the brane

- Duality between $6d$ self-dual 2-forms and SYM model

- The physical meaning of the spectral manifold (curve) and bare spectral curve *per se* – without reference to brane configuration

Hitchin systems

- Spectral curve

- SW differential

- Lax operator

- Brane configuration

Relation to conventional τ-function

- The general problem: Hitchin *vs.* τ

- Fermionic representation

- Free field representations

Dell from Casimir operators

- Quantization of integrable Hamiltonians

- Identification of the relevant groups, Casimir functions, orbits, reductions

- Elliptic algebras

- Adelization

momentum \ coordinate	rational	trigonometric	elliptic
rational	rational Calogero	trigonometric Calogero	elliptic Calogero
trigonometric	rational Ruijsenaars	trigonometric Ruijsenaars	elliptic Ruijsenaars
elliptic	dual Calogero	dual Ruijsenaars	Dell system

Figure 4: Action of the coordinate-momentum duality on the Calogero-Ruijsenaars-Dell family. Hooked arrows mark self-dual systems. The duality leaves the coupling constant g intact.

Duality

- Formulation of the coordinate-momentum duality, relation to generic canonical transformations – see Fig.4
- Duality transforms of the wave functions (zonal spherical functions)
- The study of the duals to the elliptic Calogero and Ruijsenaars models (the elliptic-rational and elliptic-trigonometric systems)
- p, q vs. p^{Jac}, a vs. "separated" variables $\mathcal{P}$, $\mathcal{Q}$
- Explicit construction of Dell systems for $SU(N)$ and other groups
- Commuting θ-functions [4]
- Dual systems [45, 3, 4] from zeroes of τ-functions [46]
- Transformation of prepotentials under duality
- Relation to the mirror symmetry
- Relation to Langlands correspondence

Theory of prepotentials (quasiclassical τ-functions)

- Definition in internal terms
- The meaning of the (generalized) WDVV equations [47]
- Further generalization of WDVV equations

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- RG flows as Whitham dynamics [25]

- Relation to (generalized) τ-functions, *i.e.* the pure group-theory objects

- Prepotentials from quantum affine algebras

- Relation to polylogarithms

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